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SPECTRA OF WAVELET TRANSFORMS

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ABSTRACT

The basic definitions of the wavelets theory are presented. Proposed is transformation which combines two transformations: wavelet and Fourier. It is compared with the well known composition of Hilbert and Fourier transformation. The properties of the new transformation and its exemplary application is presented.

INTRODUCTION

Wavelet methods [1] [2] have become well known and useful tool for various signal processing applications. They allow to achieve the time-frequency analysis of functions or vectors representing the real-world signals. There are three types of wavelet tools discussed in the literature: the continuous wavelet transform (CWT), the wavelet series (WS) and the discrete wavelet transform (DWT). The construction of each of them is based on the existence of the pair of wavelet functions, φ and ψ, which are linked together. The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2017 to Yves Meyer "for his pivotal role in the development of the mathematical theory of wavelets".

CWT and WS are defined for time domain signals based on $L^2(\Re)$ inner product, which could be treated as the measurement of the similarities between signals and wavelet functions. WS enables to represent (with high precision) each signal, $s \in L^2(\Re)$, as a sequence of real numbers.

In digital signal processing systems it is common practice that only sampled and approximate values of signals are available. In such situations more appropriate approach than CWT or WS is DWT which is adopted to sequences from $l^2(\mathbb{Z})$ space.

For WS and DWT approaches exist very efficient algorithm, known as the Mallat's pyramid algorithm [3], for finding series coefficients and discrete transform values. It gives the formula for the coefficients from lower resolution levels of WS, based on the values from higher level. This means that initial coefficients, for some high resolution level, need to be calculated first, i.e. the initialization step has to be done. For DWT this preprocessing is often reduced to taking samples of the discrete signal as the inner product values for the highest resolution level.

In this paper we focus our attention on the real valued functions from $L^2(\Re)$ and sequences from $l^2(\mathbb{Z})$ space (which we call the analog and digital signals respectively).

Engineers treat CWT as a supplementation of Fourier transform by considering kernels with practically (not theoretically because of Heisenberg uncertainty principle) compact supports simultaneously in time and frequency domain. This leads to wavelet transform where two parametric kernels for a new domain are used allowing to localize given function both in time and frequency domain.

Fourier transform is the most frequently used transformation in signal processing. The main reason is the utility of frequency representation. It is also important that mathematical models in the frequency domain are easier to use when comparing with models in the time domain. This second feature is not attainable from wavelet methods. They were created in the late twentieth century as an addition to a well known, since the first half of the nineteenth century, classical global frequency analysis methods. Wavelet transform allows a local frequency analysis of signals. It enables to track how the frequency distribution changes over time. Short-Time Fourier Transform (STFT) has similar applications and enable to analyze the successive parts of signal. This historically older method has some disadvantages when compared with the wavelet methods. The most important is the lack of automatic adjustment of the length of the analysis window to the analyzed frequency.

CONTINUOUS WAVELET TRANSFORM

Wavelets provides a new type of function representation enabling local signal analysis. Wavelet analysis uses two functions, which are inseparable pair. One of them, denoted by φ, represents low frequencies. Its average value is different from zero. The second, denoted by ψ, represents higher frequencies and its average value is equal to zero (see Fig.1).

Let us assume that some signal is represented by function $s(t)$. To examine how the distribution of frequencies changes in this signal, the wavelet transform

$$\tilde{s}_{\psi}(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} s(t)\psi\left(\frac{t-b}{a}\right) dt \tag{1}$$

can be used, where ψ is an arbitrarily selected wavelet function. Signal s is presented as a function of time t and, after transformation, we obtain the function which has two arguments. Variable b is the time in which surroundings the frequency properties of signal $s(t)$ are examined locally. The new variable, a, represents some frequency band and is proportional to the inverse of the center frequency of ψ, for the given value a. This means that transform (1) shows how strong are the frequencies surrounding the frequency proportional to $1/a$ in the analyzed signal locally around time b.

WAVELET SERIES

The wavelet decomposition is an useful tool applicable in the signal analysis. For the assumed resolution level m, the basis functions

$$\varphi_{m,n}(t) = \sqrt{2^m}\varphi(2^m t - n) \tag{2}$$

are created from function φ shifted in time t, where m and n are integer numbers. The wavelet series have form

$$s_m(t) = \sum_n c_{m,n}\varphi_{m,n}(t), \tag{3}$$

where coefficients

$$c_{m,n} = \int_{-\infty}^{\infty} s_m(t)\varphi_{m,n}(t)dt, \tag{4}$$

are inner products, since (2) functions constitute an orthonormal base.

Some wavelets have the compact supports in the frequency domain and then parameter m determines the maximal frequency in signal $s_m(t)$. Engineers treat this observation as valid always,

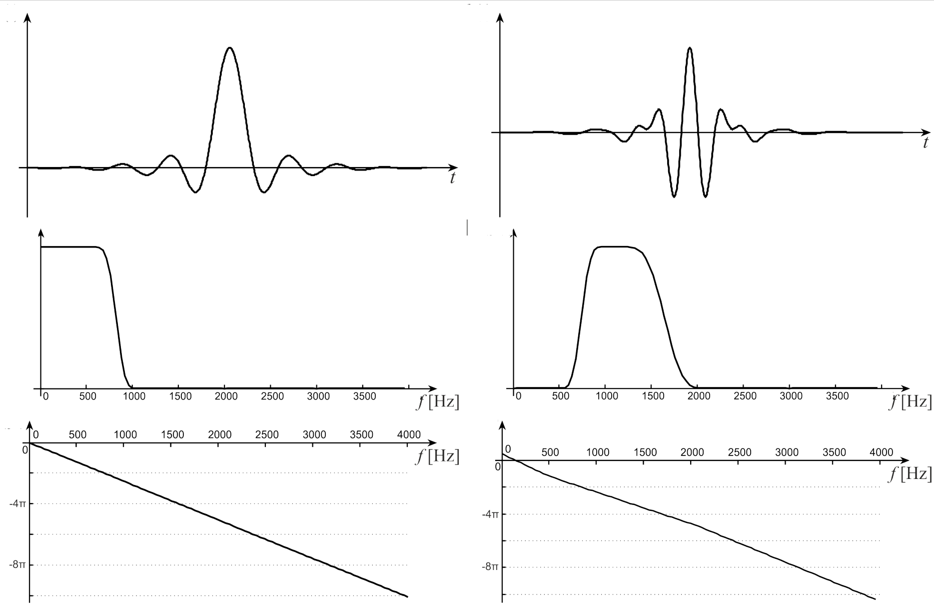

Figure 1. Meyer [2] scale function φ (left) and wavelet function ψ (right), below are their amplitude and phase spectra

because even if the spectra are not compact, a far greater part of energy is concentrated in the finite frequency band.

Each signal from the resolution level m can be decomposed into two signals, i.e.

$$s_m(t) = s_{m-1}(t) + w_{m-1}(t). \tag{5}$$

The first component s_{m-1} is treated as a coarse approximation of the signal s_m from a higher resolution level and may be represented by series

$$s_{m-1}(t) = \sum_n c_{m-1,n}\varphi_{m-1,n}(t). \tag{6}$$

The second component contains a so-called the higher resolution details and can be represented similarly, but by different base functions

$$w_{m-1}(t) = \sum_n d_{m-1,n}\psi_{m-1,n}(t). \tag{7}$$

Vector $c_m = [c_{m,n}]_n$ is a complete representation of function $s_m(t)$ hence from (3)-(6) results the vector $c_{m-1} = [c_{m-1,n}]_n$ dependence from c_m. The relationship between them has form

$$c_{m-1,n} = \sum_k h_{k-2n}c_{m,k}. \tag{8}$$

Similar consideration leads to

$$d_{m-1,n} = \sum_k g_{k-2n}c_{m,k}. \tag{9}$$

The values of coefficients h_g and g_k depend on assumed wavelet functions φ and ψ. Formulas (8) and (9) form the so-called Mallat's pyramid algorithm [3].

DISCRETE WAVELET TRANSFORM

Function $s(t)$ transformed by (1) is called an analog signal. Numerical calculations need its approximate values in evenly distributed moments of time $[n\Delta t]_n$, thus we obtain vector $[s(n\Delta t)]_n$, called a digital signal. To compute the discrete wavelet transform, we should use some method for numerical computing of integral (1). To provide quick numerical solution to this problem a very simple method was introduced. It utilizes dependencies defined for wavelet series and has been widely accepted. At the beginning the substitution

$$c_{m,n} \Leftarrow s_m(n\Delta t) \tag{10}$$

is used, and then using (8) and (9) we obtain values of Discrete Wavelet Transform

$$DWT = \left\{ \{d_{m-1,n}\}_n, \{d_{m-2,n}\}_n, \ldots, \{d_{m-M,n}\}_n, \{c_{m-M,n}\}_n \right\}. \tag{11}$$

WAVELET CRIME

The relationships (8) and (9) show the very efficient procedure for generating coefficients of DWT. However, to start this algorithm one needs to determine the initial sequence $[c_{m,n}]_n$, for some (sufficiently large) resolution level m. The problem is that coefficients (4) can not be precisely calculated, since in practice we know the signal only in discrete time points. Common practice, in such a situation, is to use sampled values (10) of signal as the initial coefficients in DWT. Such approach is commonly used, despite the fact that it was charged by Strang and Nguyen to be a "wavelet crime" [4]. Several alternative solutions to the problem have been proposed in the same paper [4]. One of them is the approach based on the Nyquist-Shannon sampling theorem which states that any band limited function can be perfectly reconstructed from a countable sequence of samples, i.e.

$$s(t) = \sum_{n\in\mathbb{Z}} s(n\Delta t) \cdot \mathrm{sinc}\Big(\frac{\pi t}{\Delta t} - \pi n\Big), \tag{12}$$

for sufficiently small sampling rate Δt. DWT can be then employed to reconstructed function defined on some interval. This approach has been analyzed and tested by Abry and Flandrin [5] and by Zhang et al. in [6].

Qian and Francis [7] have shown that starting directly from sampled values can lead to a large error. They gave an optimal filter for initialization, however the sequence obtained as a result is not easily realizable. There are several others papers dedicated to the initialization problem, such as [8] and [9].

Committing the wavelet crime (as in this paper) may be justified, if the signal is regular enough.

Theorem 1 (Frazier, [10]). *Lets suppose that:*
(1) $s(t) \in L^2(\Re)$ *and* $\varphi(t) \in L(\Re) \cap L^2(\Re)$,
(2) $\int_{-\infty}^{\infty} \varphi^2(t)\, dt = 1$,
(3) $s(t)$ *satisfy Lipschitz condition with constant L,*
(4) $C := \int_{-\infty}^{\infty} |t\varphi(t)|\, dt < \infty$.
Then we have

$$\big|\, \langle s, \varphi_{m,n}\rangle - s_m(n\Delta t)\big| \le C \cdot L \cdot 2^{-\frac{3m}{2}}. \tag{13}$$

Taking measurements of signal satisfying the assumption (1)–(4), we can compute the upper estimation of error in a suitably small intervals of time, and we have therefore the guarantee to obtain limited errors in the sequence of coefficients $[c_{m,n}]_n$.

FOURIER-WAVELET TRANSFORM

Combination of Fourier and Wavelet Transform (FWT) let us define in a following way

$$\hat{\tilde{s}}_\psi(a,f) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t)\psi\left(\frac{t-b}{a}\right) dt\, e^{-2\pi j f b} db. \tag{14}$$

FWT allows to provide specific kind of signal analysis. Using elementary properties of both used transformations, we can determine three properties of (14):

1) conservation of energy in such sense that

$$\int_{-\infty}^{\infty} s^2(t)dt = \frac{1}{\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{\tilde{s}}_\psi(a,f)|^2 \frac{dadf}{a^2} \text{ where } \eta = \int_0^{\infty} \frac{|\hat{\psi}(f)|^2}{f} df < \infty, \tag{15}$$

2) for time-shifted signal $s(t-\tau)$ we obtain the FWT in form $\hat{\tilde{s}}_\psi(a,f)e^{-2\pi j\tau f}$,

3) for the scaled signal $s(\gamma t)$ we obtain $|\gamma|^{-3/2}\hat{\tilde{s}}_\psi(a\gamma, f/\gamma)$.

Using the property that Fourier transformation preserves scalar products, equation (1) can be transformed to

$$\hat{\tilde{s}}_\psi(a,f) = \sqrt{a}\hat{\psi}^*(af)\,\hat{s}(f). \tag{16}$$

This formula means filtering of signal s. The frequency characteristic of filter depends on function conjugated to the wavelet spectrum $\hat{\psi}(af)$. Such characteristic has compact support (as for Meyer wavelets) or clearly focuses over some range of frequencies. Scaling parameter, a, shifts the spectrum and thus $\hat{\tilde{s}}_\psi(a,f)$ for assumed a is a modified portion of the signal spectrum $\hat{s}(f)$.

The link between filtration and Hilbert transformation

$$\check{s} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{s(t)}{\tau - t}dt, \tag{17}$$

is even more evident, because (17) has the form of convolution and thus Fourier transformation gives form

$$\hat{\check{s}} = -j\text{sgn}(f)\hat{s}(f). \tag{18}$$

Filter for this case has amplitude characteristic

$$A(f) = 1, \tag{19}$$

and phase characteristic

$$\theta(f) = \begin{cases} 0.5\pi & \text{for} \quad f < 0 \\ -0.5\pi & \text{for} \quad f > 0, \end{cases} \tag{20}$$

what means that this is the all-pass filter.

The first step of numerical calculations is to determine the value (11) for the digital signal s. Then, for the variable n, Fourier transformation should be used separately for each resolution level. Applying the standard Fast Fourier Transform (FFT) software, we get finally Discrete Wavelet-Fourier Transform

$$DFWT = \left\{\left\{\hat{d}_{m,k}\right\}_k, \left\{\hat{d}_{m-1,k}\right\}_k, \ldots, \left\{\hat{d}_{m-M,k}\right\}_k, \left\{\hat{c}_{m-M,k}\right\}_k\right\}. \tag{21}$$

WAVELET-FOURIER TRANSFORM

Because Fourier and wavelet transforms are not commuting operations, we obtain a separate method of analysis by determining the wavelet transform for the signal spectrum. Discrete Wavelet-Fourier Transform (DWFT) is therefore a two-step operation. First, for discrete signal, $s \in \Re^N$, the discrete spectrum $\hat{s} \in C^N$ is determined using FFT. Then for this discrete spectrum DWT is computed.

EXAMPLE AND CONCLUSIONS

Wavelet theory provides researchers and engineers with a powerful tool for time-frequency signal analysis. Successful applications of the wavelet methods include biomedical signal analysis, image and audio processing, technical measurement analysis and other time-series processing areas. Proposed hybrid DFWT and DWFT are relatively new techniques. Further research should focus on their features, and new application areas should be proposed for both of them.

This paper presented theoretical basics of the wavelet transform, and it's corresponding discrete version - namely discrete wavelet decomposition. The concept of the wavelet crime was

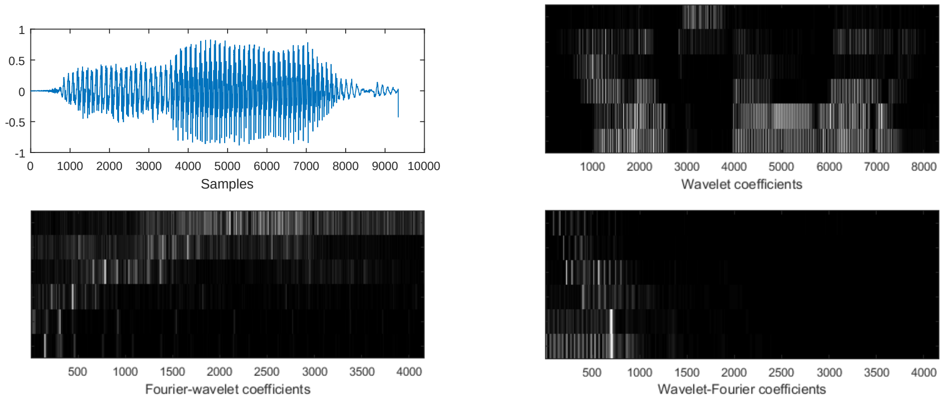

Figure 2. Speech signal presented in various domains: time waveform (left-upper), DWT (right-upper), amplitude of a DFWT (left-bottom), and amplitude of a DFWT (right-bottom). Upper rows of each wavelet spectrum represent higher resolution levels.

introduced and discussed as well. Finally, hybrid Fourier-wavelet and wavelet-Fourier transforms were defined in both continuous and discrete domains.

Exemplary plots of discrete transforms: wavelet, Fourier-wavelet and wavelet-Fourier for a short speech signal are presented in Fig.2. The signal is an isolated phoneme /a/ produced by a male speaker. A 5-level wavelet transform was computed using a Meyer decomposition filters. One can observe that DFWT and DWFT produce entirely different results.

Frequency analysis is the most commonly used tool in a wide variety of signal processing applications. These methods include not only the Fourier transform but also: cosine transformation, mel-frequency cepstral coefficients, as well as the mentioned above STFT, CWT, DWT. It can be expected that the proposed DFWT and DWFT transformations will also become useful tools. The authors examine their capabilities in speech technology, i.e. speech recognition [11] and [12].

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