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CRISS-CROSS MODEL OF TUBERCULOSIS FOR HOMELESS AND NON-HOMELESS SUBPOPULATIONS

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ABSTRACT

We present a criss-cross model describing tuberculosis epidemic dynamics. The case in study considers Warmian-Masurian province of Poland and is related to actions of active detecting of tuberculosis in homeless people subpopulation. Therefore, the whole population is divided into two subpopulations: non-homeless and homeless people. Each of these subpopulations consists of two groups – susceptible and infected people. We present the analysis of existence and stability of stationary states. On the basis of this analysis we conclude that in many cases at least one of the subpopulations becomes extinct, and only for specific parameter values coexistence of both subpopulations is possible.

INTRODUCTION

As National Tuberculosis and Lung Diseases Research Institute in Poland reports, in our country in the years 2003-2012 the incidence of pulmonary tuberculosis declined by 26%. The drop was especially noticeable in Warmian-Masurian province, where it was equal to 53%. In the community of homeless people in this region, four programs of active detection of tuberculosis were conducted in the years 2004-2011. As a result, the incidence of tuberculosis dropped not only among homeless individuals, but also among the whole population in the region. The outcomes of active case finding campaigns and the model describing the dynamics of tuberculosis in the population were described in [4].

In this paper, following [4], we present simple criss-cross model of the spread of tuberculosis. Such type of model are often used in the description of epidemics spreading not only between different subpopulations of one population, but also among different populations; cf. [2]. We consider two subpopulations: non-homeless and homeless people. Let us divide each of these subpopulations into two groups: susceptible and infected people. The illness is transmitted not only among one subpopulation, but also from the homeless people to non-homeless ones and vice versa. That is why the criss-cross model is used. We follow the notation of [4]:

- $S_1(t)$ is the number of the non-homeless susceptible people at time t,
- $S_2(t)$ is the number of the homeless susceptible people at time t,
- $I_1(t)$ is the number of the non-homeless infectious people at time t,
- $I_2(t)$ is the number of the homeless infectious people at time t.

Let us denote

$$N_1(t) = S_1(t) + I_1(t), \qquad N_2(t) = S_2(t) + I_2(t), \tag{1}$$

i.e. $N_1(t)$ means the number of the non-homeless people at time t and $N_2(t)$ is the number of the homeless people at time t. In the following we use the notation with subscript i, $i = 1, 2$, and moreover we write S_i, I_i and N_i instead of $S_i(t)$, $I_i(t)$ and $N_i(t)$, accordingly.

The dynamics of tuberculosis epidemic can be described (cf. [4]) by the system of four equations

$$\dot{S}_1 = -\beta_{11}S_1I_1 - \beta_{12}S_1I_2 + \gamma_1 I_1 + A_1 S_1, \tag{2a}$$

$$\dot{I}_1 = \beta_{11}S_1I_1 + \beta_{12}S_1I_2 - (\gamma_1 + \alpha_1 - A_1)I_1, \tag{2b}$$

$$\dot{S}_2 = -\beta_{22}S_2I_2 - \beta_{21}S_2I_1 + \gamma_2 I_2 + A_2 S_2, \tag{2c}$$

$$\dot{I}_2 = \beta_{22}S_2I_2 + \beta_{21}S_2I_1 - (\gamma_2 + \alpha_2 - A_2)I_2, \tag{2d}$$

where β_{11}, β_{22}, β_{12}, β_{21} are transmission coefficients between non-homeless people, between homeless people, from homeless people to non-homeless ones and from non-homeless people to homeless ones, respectively. In this paper we assume that these coefficients are fixed and positive. Parameters α_1 and α_2 stand for disease-related death rates for the non-homeless and homeless people, accordingly, γ_1, γ_2 are recovery coefficients for the non-homeless and homeless people, respectively. The constants α_i and γ_i are non-negative. Additionally, A_1 is a fixed net birth rate for the non-homeless people. Therefore, $A_1 = b_1 - d_1$, where b_1 means a birth coefficient and d_1 is a natural death coefficient, while A_2 stands for a constant which includes net reproduction for the homeless people and migration to the subpopulation of the homeless people.

BASIC PROPERTIES OF THE MODEL

The right-hand sides of Eqs. (2) are function of $\mathbf{C}^1$ class, so they fulfil local Lipschitz conditions. By Picard's existence and uniqueness theorem, solutions of the system exist and are unique for any initial data.

Assume now that $S_i(0), I_i(0) > 0$. Then, until $S_1(t), I_1(t) > 0$, we have

$$\dot{S}_1 \geqslant -\beta_{11}I_1S_1 - \beta_{12}I_2S_1 + A_1S_1,$$

yielding

$$\frac{\dot{S}_1}{S_1} \geqslant -\beta_{11}I_1 - \beta_{12}I_2 + A_1.$$

From Chaplygin-Perron's differential inequalities theorem [3] we get

$$S_1(t) \geqslant S_1(0)\exp\int_0^t (-\beta_{11}I_1(s) - \beta_{12}I_2(s) + A_1)\,ds > 0,$$

which means that $S_1(t)$ is positive for every $t > 0$ for which the solution exists. The analogous reasoning conducted for Eq. (2c) gives positiveness of $S_2(t)$, $t > 0$. Similarly, we can prove that $I_1(t) > 0$ and $I_2(t) > 0$, $t > 0$.

In the following, we shall show that solutions exist for all $t > 0$. Adding Eqs. (2a) and (2b) or Eqs. (2c) and (2d) we get

$$\dot{S}_i + \dot{I}_i = A_i(S_i + I_i) - \alpha_i I_i. \tag{3}$$

By the notation (1) we can re-write Eq. (3) in the form

$$\dot{N}_i = A_iN_i - \alpha_i I_i. \tag{4}$$

Since $I_i(t)$ is non-negative, we obtain

$$\dot{N}_i \leqslant A_iN_i,$$

and therefore

$$N_i(t) \leqslant N_i(0)\mathrm{e}^{A_i t}.$$

This means that the growth of solutions of (2) is at most exponential, and hence $S_1(t)$, $S_2(t)$, $I_1(t)$ and $I_2(t)$ are defined for every $t \geqslant 0$.

ANALYSIS OF STATIONARY STATES

Let us denote

$$k_i = \gamma_i + \alpha_i - A_i, \qquad \kappa_i = \frac{\alpha_i}{A_i} - 1, \;\; A_i \neq 0, \qquad i = 1, 2.$$

Notice that $\kappa_i \geqslant 0$ implies $\alpha_i \geqslant A_i > 0$ and $k_i \geqslant 0$, while if $A_i < 0$, then $\kappa_i < 0$ and $k_i > 0$.

Existence of stationary states

Looking for stationary states we replace the left-hand side of Eq. (4) by zero:

$$0 = A_i N_i - \alpha_i I_i. \tag{5}$$

Putting (1) into (5) gives

$$0 = A_i(I_i + S_i) - \alpha_i I_i = (A_i - \alpha_i)I_i + A_i S_i.$$

Hence

$$S_i = \frac{(\alpha_i - A_i)I_i}{A_i} = \kappa_i I_i, \tag{6}$$

where $A_i \neq 0$. Notice that a zero stationary state, indicated by $E_0 := (0, 0, 0, 0)$, always exists, regardless of the model parameters. The condition $\kappa_i < 0$ results in $S_i \cdot I_i < 0$, which is impossible due to the meaning of the variables. For $\alpha_i \neq A_i$ we state that if $\kappa_i = 0$, then $S_i = 0$, but this immediately gives $I_i = 0$. Assume now $\kappa_1 > 0$. For $I_2 = 0$, Eq. (2a) yields

$$-\beta_{11} S_1 + \gamma_1 + A_1 \kappa_1 = 0.$$

Hence, there exists the semi-positive stationary solution $E_1 = (S_1^*, I_1^*, 0, 0)$ with $S_1^* = \frac{k_1}{\beta_{11}}$. By symmetry, the stationary solution $E_2 = (0, 0, S_2^*, I_2^*)$, $S_2^* = \frac{k_2}{\beta_{22}}$, exists for $\kappa_2 > 0$. These equilibria represent the situation when only one of the subpopulations, the non-homeless people or the homeless ones, exists.

Let us denote also a non-zero stationary solution as E_+:

$$E_+ := (\bar{S}_1, \bar{I}_1, \bar{S}_2, \bar{I}_2), \qquad \bar{S}_1,\ \bar{I}_1,\ \bar{S}_2,\ \bar{I}_2 > 0.$$

Under certain existence conditions we obtain the specific formulae for $\bar{I}_i$:

$$\bar{I}_1 = \frac{k_2\beta_{12}\kappa_1 - k_1\beta_{22}\kappa_2}{(\beta_{12}\beta_{21} - \beta_{11}\beta_{22})\,\kappa_1\kappa_2}, \quad \bar{I}_2 = \frac{k_1\beta_{21}\kappa_2 - k_2\beta_{11}\kappa_1}{(\beta_{12}\beta_{21} - \beta_{11}\beta_{22})\,\kappa_1\kappa_2},$$

where $(\beta_{12}\beta_{21} - \beta_{11}\beta_{22}) \neq 0$.

Analysing the sign of κ_i, we can formulate the necessary conditions for the existence of stationary states:

- if κ_1, $\kappa_2 \leqslant 0$, then only E_0 exists,
- if $\kappa_2 \leqslant 0$ and $\kappa_1 > 0$, then E_+ and E_2 do not exist, but E_1 exists,
- if $\kappa_1 \leqslant 0$ and $\kappa_2 > 0$, then E_+ and E_1 do not exist, but E_2 exists,
- if κ_1, $\kappa_2 > 0$, then E_0, E_1 and E_2 exist, while the existence of E_+ requires additional conditions to be satisfied.

Furthermore, the additional conditions for the existence of E_+ read

$$k_1\kappa_2\frac{\beta_{21}}{\beta_{11}} < k_2\kappa_1 < k_1\kappa_2\frac{\beta_{22}}{\beta_{12}} \tag{7}$$

or

$$k_1\kappa_2\frac{\beta_{21}}{\beta_{11}} > k_2\kappa_1 > k_1\kappa_2\frac{\beta_{22}}{\beta_{12}}.$$

Stability of stationary states

Investigating the stability, we calculate a Jacobian matrix of System (2). This matrix, indicated by J, has the form

$$J = \begin{pmatrix} -\beta_{11}I_1 - \beta_{12}I_2 + A_1 & -\beta_{11}S_1 + \gamma_1 & 0 & -\beta_{12}S_1 \\ \beta_{11}I_1 + \beta_{12}I_2 & \beta_{11}S_1 - k_1 & 0 & \beta_{12}S_1 \\ 0 & -\beta_{21}S_2 & -\beta_{22}I_2 - \beta_{21}I_1 + A_2 & -\beta_{22}S_2 + \gamma_2 \\ 0 & \beta_{21}S_2 & \beta_{22}I_2 + \beta_{21}I_1 & \beta_{22}S_2 - k_2 \end{pmatrix}.$$

Let us first focus on the zero state. The Jacobian matrix for this state reads

$$J(E_0) = \begin{pmatrix} A_1 & \gamma_1 & 0 & 0 \\ 0 & -k_1 & 0 & 0 \\ 0 & 0 & A_2 & \gamma_2 \\ 0 & 0 & 0 & -k_2 \end{pmatrix}.$$

The eigenvalues of this matrix are equal to $\lambda_1 = A_1$, $\lambda_2 = -k_1$, $\lambda_3 = A_2$, $\lambda_4 = -k_2$. Let us suppose A_1, $A_2 < 0$. Each of the eigenvalues is negative, which gives the local stability of E_0. Moreover, the inequalities $\dot{N}_i = A_iN_i - \alpha_iI_i < 0$ for $i = 1$ and $i = 2$ result in the global stability of E_0. If at least one parameter A_1 or A_2 is positive, then at least one of the eigenvalues is positive, which gives the instability of E_0.

Now we investigate the stability of $E_2 = (0, 0, S_2^*, I_2^*)$ with the necessary condition $\kappa_2 > 0$. The characteristic equation for this solution reads

$$\det \begin{pmatrix} -\beta_{12}I_2^* + A_1 - \lambda & \gamma_1 & 0 & 0 \\ \beta_{12}I_2^* & -k_1 - \lambda & 0 & 0 \\ 0 & -\beta_{21}S_2^* & -\frac{\gamma_2}{\kappa_2} - \lambda & A_2 - \alpha_2 \\ 0 & \beta_{21}S_2^* & \frac{k_2}{\kappa_2} & -\lambda \end{pmatrix} = 0.$$

Hence we obtain

$$\lambda^2 - \operatorname{tr} M_1 \cdot \lambda + \det M_1 = 0 \quad \text{or} \quad \lambda^2 - \operatorname{tr} M_2 \cdot \lambda + \det M_2 = 0,$$

where

$$M_1 = \begin{pmatrix} -\frac{\gamma_2}{\kappa_2} & A_2 - \alpha_2 \\ \frac{k_2}{\kappa_2} & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -\beta_{12}I_2^* + A_1 & \gamma_1 \\ \beta_{12}I_2^* & -k_1 \end{pmatrix}.$$

The conditions for stability of E_2 are $\det M_i > 0$ and $\operatorname{tr} M_i < 0$ for $i = 1, 2$. Obviously, as $\kappa_2 > 0$, these conditions are satisfied for M_1. For M_2 we obtain

$$(k_1 - \gamma_1)\,\beta_{12}I_2^* - A_1k_1 > 0 \tag{8}$$

and

$$k_1 + \beta_{12}I_2^* - A_1 > 0. \tag{9}$$

Notice that for $A_1 \neq 0$, Ineq. (8) is equivalent to $A_1\,(\kappa_1\beta_{12}I_2^* - k_1) > 0$. Hence, if $A_1 < 0$, then both Ineqs. (8) and (9) are satisfied. Moreover, for $A_1 = 0$ they are satisfied as well. Now, consider $A_1 > 0$. Then from Ineq. (9) we have $A_1 < k_1 + \beta_{12}I_2^*$, while Ineq. (8) implies $k_1 < \kappa_1\beta_{12}I_2^*$.

Therefore, the stationary state E_2 is stable if

- either $A_1 \leqslant 0$
- or $0 < A_1 < k_1 + \beta_{12}I_2^* < (\kappa_1 + 1)\beta_{12}I_2^*$.

Because of the symmetry of Eqs. (2), the stability conditions for E_1 are analogous to those for E_2.

Conditions for stability of the positive stationary state are the subject of our present study. However, using Routh-Hurwitz Criterion it is easy to see that Ineqs. (7) are necessary conditions of stability.

In order to prove the global stability of the appropriate equilibrium of the system (2), Lyapunov function can be applied. However, in general it is hard to find the Lyapunov function for the particular epidemiological model. This technique for the simple ones is used in [1].

Numerical simulations

In this section we complement analytical results presented above with some numerical results. First, exemplary simulation for the stationary solution E_+ was conducted. The plots showing dependence of the solution of Eqs. (2) on time for parameters $\alpha_1 = \alpha_2 = 0.09$, $\gamma_1 = \gamma_2 = 0.91$, $A_1 = A_2 = 0.01$, $\beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0.5$ are presented in Fig. 1. The coefficients α_1, α_2, γ_1, γ_2 were estimated on the basis of statistical yearbooks from the Central Statistical Office of Poland. The parameters A_1, A_2, β_{11}, β_{22}, β_{12} and β_{21} were chosen arbitrary. The values of variables $(S_1(0), I_1(0), S_2(0), I_2(0)) = (1465775, 390, 715, 7)$, used as the initial condition, were taken from National Tuberculosis and Lung Diseases Research Institute. The numbers $S_1(0)$, $I_1(0)$, $S_2(0)$ and $I_2(0)$ describe the sizes of the subpopulations in Warmian-Masurian province of Poland in the year 2001.

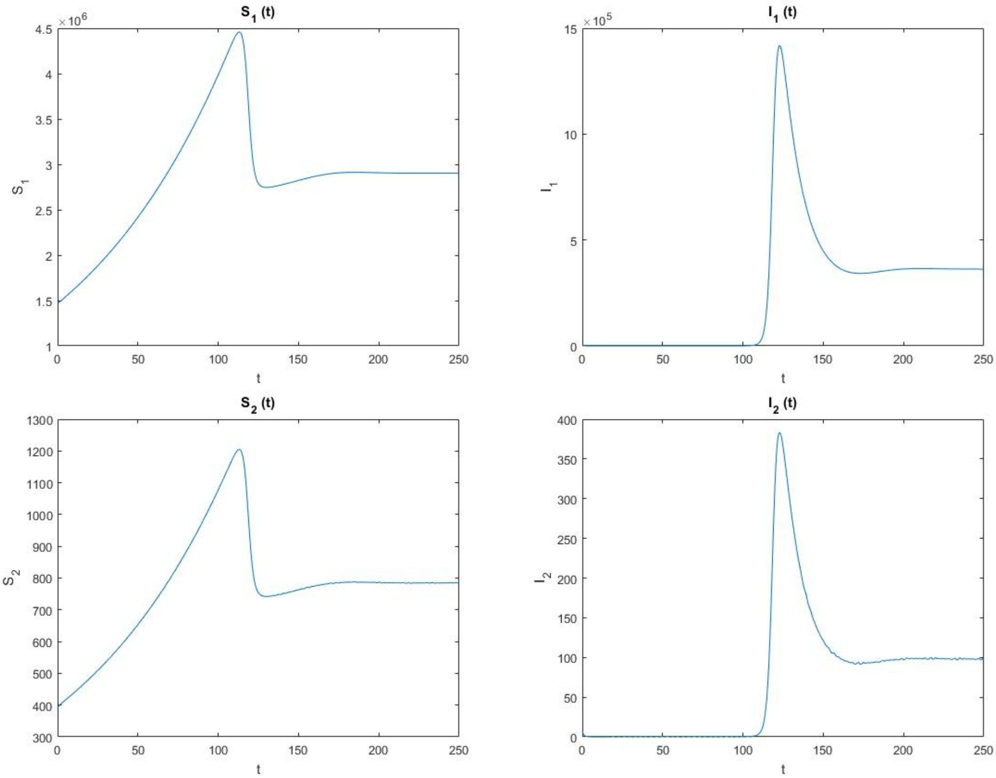

Figure 1. Plots of S_1, I_1, S_2 and I_2 illustrating stability of E_+.

Next, the sample simulation for E_1 stability was also done. The phase portraits in (S_1, I_1), (S_2, I_2), (S_1, S_2) and (I_1, I_2) planes are presented in Fig. 2. The parameters: $\alpha_1 = \alpha_2 = 0.09$, $\gamma_1 = \gamma_2 = 0.91$, $A_1 = 0.01$, $A_2 = -0.01$, $\beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0.5$ were taken. In Fig. 2 the exemplary initial condition $(S_1(0), I_1(0), S_2(0), I_2(0)) = (100, 20, 10, 5)$ was assumed.

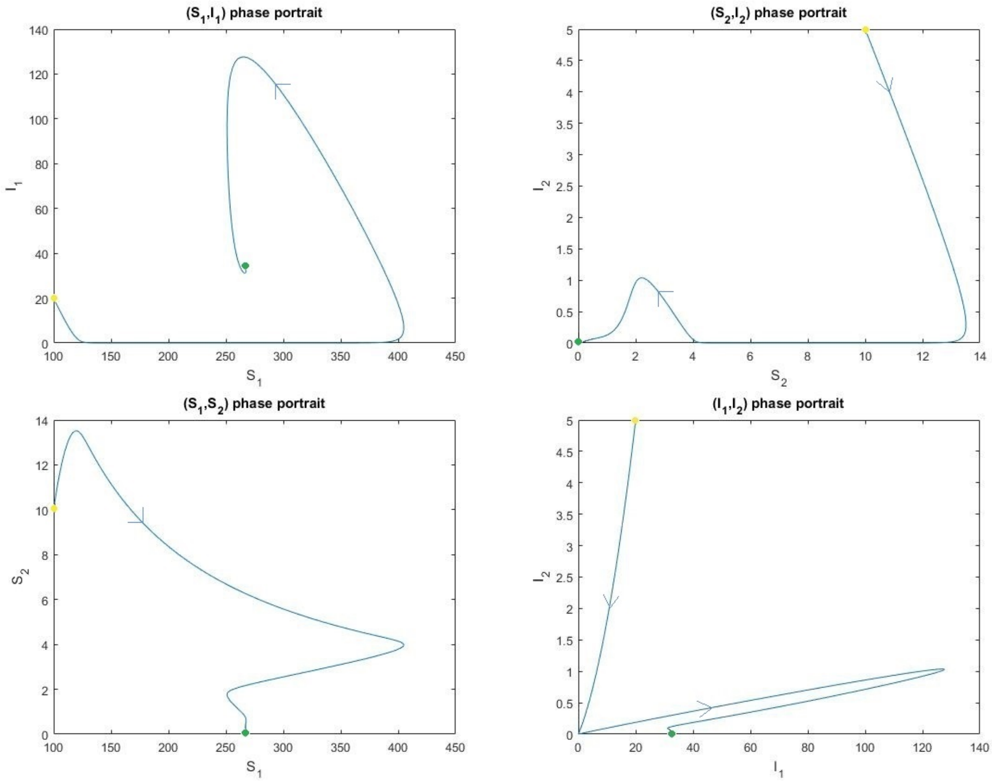

Figure 2. Phase portraits for (S_1, I_1), (S_2, I_2), (S_1, S_2) and (I_1, I_2) planes. The initial condition is marked with the yellow circle, the stationary states is indicated by the green one.

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