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TIME DELAYS AND THE GOTTMAN, MURRAY ET AL. MODEL OF MARITAL INTERACTIONS

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ABSTRACT

In this article we consider the Gottman, Murray et al. model of marital interactions. We focus on the analysis of the influence of delays into the model dynamics. It occurs that splitting the attention into the previous and present round of talks between spouses does not influence the stability condition given by Murray.

INTRODUCTION

Mathematical modelling of emotions of communicated persons, including couples like marriages and their emotional interactions has not so long history. The most commonly used tool are *dynamical systems*, both continuous and discrete. Probably the first who tried to use dynamical systems in that context was Steven Strogatz, who in 1988 published one-page article focused on it [9]. Basic models of relationships between two persons are described in the framework of linear models [4, 9, 10]. Such simple models cannot describe the relationships precisely, but can reflect some key features of them. On the other hand, nonlinear models can bring some insight into such models dynamics, and moreover could have interesting interpretation [3, 7, 8].

Another issue that could be considered is the influence of delays. There are some psychological evidences (cf. discussion in [2]) that one of the partners is deliberative and his/her emotions are delayed with respect to stimulus. Hence, in this paper we follow the idea presented in [2] (in the context of linear models) and [1] (in the context of nonlinear models) and consider the influence of delays into marital interactions in the context of discrete dynamical systems used by Gottman, Murray et al. J. M. Gottman, in his pioneering studies on the usage of mathematical modelling of marital interactions and predicting divorces on the basis of such modelling, worked together with J. D. Murray [5]. They proposed a discrete dynamical system to describe wife and husband emotional states and reactions during the session in Gottman's clinic.

Let x_n, y_n denote intensity of emotions of wife and husband, respectively, during nth session of conversation on a given topic problematic for them. The model reads

$$\begin{aligned} x_{n+1} &= Ax_n + B + f(y_n), \\ y_{n+1} &= Cy_n + D + g(x_{n+1}), \end{aligned} \tag{1}$$

where:

- linear parts of the right-hand side of Eqs. (1) reflect so-called uninfluenced or inner emotional dynamics of each of the partners;

- the functions f and g describe their influence to each other.

Non-symmetricity of the system is a consequence of the fact that wife talks first during each round of the speech.

It is assumed that in the absence of the partner, both the wife and the husband are able to achieve her/his uninfluenced steady state:
(1) $B/(1-A)$ for the wife,
(2) $D/(1-C)$ for the husband.

Moreover the convergence to the steady state is monotonic. Therefore, it is necessary to assume

$$A \in (0,1) \quad \text{and} \quad C \in (0,1).$$

As regards the influence functions, Gottman *et al.* did not assumed any specific properties of them and their forms are just a simple consequence of a fit to the data. It occurred that it was enough to choose partially linear influence functions to obtained sufficiently good fit with the data, compare Fig. 1.

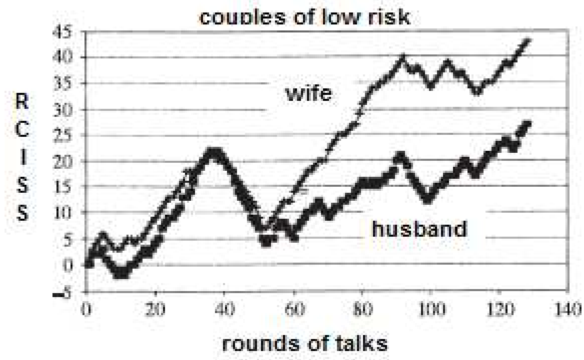

Figure 1. Example of the data from [5]. Rapid Couples Interaction Scoring System is a special system of scoring the behaviour of spouses during talks.

STABILITY ANALYSIS

Now, we turn to the model analysis. Steady states of Eqs. (1) satisfy

$$\bar{x} = A\bar{x} + B + f(\bar{y}),$$
$$\bar{y} = C\bar{y} + D + g(\bar{x}),$$

that is

$$\bar{x} = \frac{B + f(\bar{y})}{1-A}, \quad \bar{y} = \frac{D + g(\bar{x})}{1-C}. \tag{2}$$

The condition of stability given by Murray [6] reads

$$f'(\bar{y})g'(\bar{x}) < (1-A)(1-C). \tag{3}$$

This condition is associated with the location of “null-clines” proposed by Murray in the phase space (x, y). These “null-clines” are described by

$$\begin{cases} x = \frac{B+f(y)}{1-A}, & \text{for the variable } x, \\ y = \frac{D+g(x)}{1-C}, & \text{for the variable } y. \end{cases}$$

Any steady state described by Eqs. (2) lies on the intersection of these curves. It should be noticed that

- $\frac{dx}{dy} = \frac{f'(y)}{1-A}$ on the null-cline for x,
- $\frac{dy}{dx} = \frac{g'(x)}{1-C}$ on the null-cline for y,

while the condition of stability described by Eq. (3) could be rewritten as

$$\frac{g'(x)}{1-C} < \frac{1}{\frac{f'(y)}{1-A}},$$

that is the slop for x null-cline is less than the slop for y null-cline in the space (x, y); cf. Fig. 2.

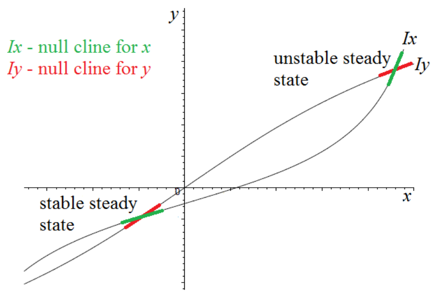

Figure 2. Example of stable and unstable steady state of Eqs. (1)

It occurs that in general the condition of stability could be different than (3).

Proposition 1. *Let A, $C \in (0, 1)$ and assume that a steady state of Eqs.* (1) *exists. Then if*

$$-(1 + A)(1 + C) < f'(\bar{y})g'(\bar{x}) < (1 - A)(1 - C),$$

then this steady state is asymptotically stable.

Proof. Calculating a characteristic polynomial of Eqs. (1) one gets

$$W(\lambda) = \lambda^2 - (A + C + g'(\bar{x})f'(\bar{y}))\lambda + AC, \tag{4}$$

and the discriminant of (4) reads

$$\Delta = (A - C)^2 + \Big(2(A + C) + g'(\bar{x})f'(\bar{y})\Big)g'(\bar{x})f'(\bar{y}).$$

Assuming that $g'(\bar{x})f'(\bar{y}) \geq 0$, as in Fig. 2, we obtain $\Delta \geq 0$, obviously. Hence, we have two positive zeros of W. Both of them have modulus less than one iff

$$\lambda_2 = \frac{A + C + g'(\bar{x})f'(\bar{y}) + \sqrt{\Delta}}{2} < 1,$$

which is equivalent to

$$\sqrt{\Delta} < 2 - \big(A + C + g'(\bar{x})f'(\bar{y})\big). \tag{5}$$

This inequality could be satisfied only when $2 - \big(A + C + g'(\bar{x})f'(\bar{y})\big) > 0$, that is $f'(\bar{y})g'(\bar{x}) < 2 - A - C$. Under this assumption we can square both sides of (5) and obtain

$$g'(\bar{x})f'(\bar{y}) < (1 - A)(1 - C) < 2 - A - C, \quad \text{for} \quad A,\ C \in (0, 1).$$

This is in fact the case studied by Murray.

Another possibility is obtained for $f'(\bar{y})g'(\bar{x}) < 0$ and $\Delta \geq 0$. Now, the inequality $g'(\bar{x})f'(\bar{y}) > -2 - A - C$ must be satisfied, and then

$$f'(\bar{y})g'(\bar{x}) > -(1 + A)(1 + C) > -2 - A - C, \quad \text{for} \quad A,\ C \in (0, 1),$$

which is not so easy to interpret as the condition (3).

The last possibility is for $\Delta < 0$. Then we have two complex zeros $\lambda_1 = \overline{\lambda_2}$ and

$$\lambda_1\lambda_2 = AC < 1.$$

This means that the stability does not depend on the model parameters in this case. □

In the next section we follow the idea of Murray which is based on the basic assumption (3) and the additional one

$$g'(\bar{x})f'(\bar{y}) \geq 0. \tag{6}$$

TIME DELAYS

Let us think about introducing the possibility of reacting not only with respect to the present round of talk, but also to the previous one.

It could be interpreted as introducing time delay into the model. In particular, the symmetric version of Eqs. (1), that is

$$\begin{aligned} x_{n+1} &= Ax_n + B + f(y_n), \\ y_{n+1} &= Cy_n + D + g(x_n), \end{aligned}$$

could be interpreted in such a way that the husband reacts with one unit delay. However, much more realistic is the situation in which responding to the wife arguments her husband also remembers the previous round of speech. Hence, we can assume that during the $(n+1)$th round of talk the husband splits his attention both into the present and previous rounds. In general, the reaction with respect to the previous round could be described by some function $g_1(x_n)$, while for the present round by $g_2(x_{n+1})$. However, it seems reasonable to assume $g_1(x) = g_2(x)$, and splitting the attention is reflected by some coefficient $\alpha \geq 0$.

Therefore, to reflect the delayed respond of the husband we can propose the following system of equations

$$\begin{aligned} x_{n+1} &= Ax_n + B + f(y_n), \\ y_{n+1} &= Cy_n + D + \alpha g(x_n) + (1-\alpha)g(x_{n+1}), \end{aligned} \tag{7}$$

where α is the parameter of "splitting". We see that:

- $\alpha = 0$ represents "pure instantaneous" reaction;
- $\alpha = 1$ represents "pure delayed" reaction.

Proposition 2. *Under the assumption* (6) *the condition of stability for Eqs.* (7) *is the same as for Eqs.* (1).

Proof. The characteristic polynomial reads

$$W_1(\lambda) = \lambda^2 - \big(A + C + (1-\alpha)f'(\bar{y})g'(\bar{x})\big)\lambda + AC - \alpha f'(\bar{y})g'(\bar{x}), \tag{8}$$

and for condition (6) satisfied, it has

- either real non-negative zeros for $AC > \alpha f'(\bar{y})g'(\bar{x})$,
- or real zeros with opposite signs, but the positive zero has greater modulus because $W_1'(0) < 0$.

One can easily check that the condition of stability is the same as in the case without delay and does not depend on the parameter α. □

Now, we consider the delayed reaction of the wife. Introducing the splitting like for the husband, we obtain the following system

$$\begin{aligned} x_{n+1} &= Ax_n + B + \alpha f(y_{n-1}) + (1-\alpha)f(y_n), \\ y_{n+1} &= Cy_n + D + g(x_{n+1}). \end{aligned} \tag{9}$$

To study stability we present Eqs. (9) in the normal form, that is with the right-hand side depending only on the nth time step. Therefore, we need to increase the dimension of our dynamical system. Let us introduce a new variable

$$(x_n, y_{n-1}, y_n),$$

and denote

$$G_1(x_n, y_{n-1}, y_n) = Ax_n + B + \alpha f(y_{n-1}) + (1-\alpha) f(y_n),$$
$$G_2(x_n, y_{n-1}, y_n) = Cy_n + D + g\Big(G_1(x_n, y_{n-1}, y_n)\Big).$$

Then the dynamics of Eqs. (9) is described by iterations of the function

$$F_w\Big(x_n, y_{n-1}, y_n\Big) = \Big(G_1(x_n, y_{n-1}, y_n), y_n, G_2(x_n, y_{n-1}, y_n)\Big). \tag{10}$$

Proposition 3. *Under the assumption* (6) *the condition of stability for Eqs.* (9) *is the same as for Eqs.* (1)*.*

Proof. The function (10) yields the characteristic matrix

$$dF_w(\bar{x}, \bar{y}) - \lambda\mathbb{I} = \begin{pmatrix} A-\lambda & \alpha f'(\bar{y}) & (1-\alpha) f'(\bar{y}) \\ 0 & -\lambda & 1 \\ Ag'(\bar{x}) & \alpha f'(\bar{y}) g'(\bar{x}) & C + (1-\alpha) f'(\bar{y}) g'(\bar{x}) - \lambda \end{pmatrix}.$$

Calculating the characteristic polynomial we see that it is equal to $-\lambda W_1(\lambda)$, where W_1 is described by (8). This means that the condition of stability remains the same again. □

DISCUSSION

In this paper we have presented analysis of the stability of the Gottman, Murray et al. model and the influence of one unit delay into this stability. It occurs that the delay does not change the condition of stability, unless the condition (6) is satisfied. There arise several questions about conditions of stability and the influence of time delays. First, we can consider the delay of more than one unit. Partial answer is known – we can show that for pure delayed reaction of the wife or husband the magnitude of delay does not play a role in the stability condition. In fact, characteristic matrices take block forms for which characteristic polynomials do not change a lot comparing to (4) or (8). On the other hand, the influence of delays in the case of splitting the husband or wife attention into several rounds of talk should be also considered.

Another possibility is to consider the case when the condition (6) is not satisfied. It can happen that changes of stability are possible. It is the interesting subject for future investigations. It should be also noticed that the assumption $A, C \in (0,1)$ which is associated with monotonic convergence to the inner emotional state is not necessarily true in general. It seems that most of us experience emotional fluctuations, so monotonic behaviour is a simplification. Therefore, another challenging topic is to combine emotional fluctuations with delayed reactions of the partners. Moreover, both partners can react react with delay simultaneously.

Last, but not least, the behaviour of non-stable couples should be considered. We plan to study this behaviour at least numerically, for the couples and parameters considered by Gottman, Murray et al. [5].

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