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A MODEL OF A GIANT TREE TRUNK

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ABSTRACT

A simple model of a tree trunk is described. Its purpose is to explain the shape of giant trees and to analyse the durability of the trunk subject to the gravitation and wind forces. The model has the form of a nonlinear integral equation, which may be solved numerically using the collocation method.

INTRODUCTION

This paper was inspired by the Giant Sequoia tree (*Sequoiadendron giganteum*) growing in Jasov, a town in eastern Slovakia. The paper attempts to explain, why the trunks of giant trees have the shape they have. The shape of any plant and its parts is the effect of the evolution, which forces plants to make the best use of available resources, like water, nutrients in the soil and the sunlight. There are also restrictions, of which here we pay attention to the mechanical durability of plant organs. The durability of wood is of primary importance for the highest trees. A simple model of a tree trunk presented in this paper is based on the assumption that during the strongest wind the tree can resist, the maximal tension in the horizontal sections at all altitudes is the same. In this way the material is used to build the tree trunk in an optimal way.

We consider only the forces caused by gravitation and wind. The model developed below is based on a number of simplifying assumptions. One of them is the linear correspondence between the tension and deformation of the material, widely known as the Hooke's law [1]. It is assumed that mechanical properties of wood are the same in the entire volume of the trunk. Only stretching and compressing the fibres along the trunk axis are considered. It is assumed that the trunk is a solid of revolution, *i.e.* each horizontal section is circular. The problem is to find the function, whose graph is the generatrix of this solid of revolution.

Other assumptions are as follows: the distribution of the weight of branches along the trunk is proportional to the distribution of the weight of the trunk itself. This assumption is of course not true, but in case of the Giant Sequoia trees, like the one in Jasov, due to the shape of the crown it seems not very distant from reality. Another assumption is about the distribution of the wind force bending the trunk. Here we assume that the force per unit length is constant along the trunk. Usually the wind speed increases with the altitude, but the highest branches are short.

The tree trunk may be bent by the wind and, if the trunk is not exactly vertical, also by the tree weight. The general shape of trunks of trees at some age is an attribute of the species. Therefore we assume that the parameter α, influencing the trunk shape, is the maximal slant for the species, and the angle of inclination of a particular tree may be some $\beta \neq \alpha$. The model makes it possible to find the tensions also in this case.

FORCES IN THE TRUNK

The calculation in this paper is done using a Cartesian system of coordinates, whose x axis is vertical (with 0 at the ground level), the y axis is horizontal and any slant of the trunk is oriented towards the positive y halfaxis. The z axis is perpendicular to the other two. To consider the maximal tensions we assume that the wind direction and orientation is that of the positive y halfaxis.

The trunk height is denoted by H. The horizontal section of the trunk at the level x is a circle $K(x)$ of radius $r(x)$. Based on the assumption that the distribution of weight of branches along the trunk is proportional to the distribution of the trunk weight (see Introduction), we can calculate the weight of the part of the tree above x as

$$S(x) = \int_x^H Dr^2(t)\,\mathrm{d}t,$$

where D is a constant. The weight is balanced by the compressing tension σ_c, constant in the area of the section, hence

$$\sigma_\mathrm{c} = \frac{S(x)}{\pi r^2(x)} = \frac{D}{\pi r^2(x)} \int_x^H r^2(t)\,\mathrm{d}t.$$

For each horizontal section $K(x)$ we introduce a coordinate system (η, ζ), whose axes are respectively parallel to the y, z axes of the system defined above. For a beam (trunk), whose section is changing slowly along its axis and which is made of a material satisfying the Hooke's law, the tension caused by bending in the xy plane at the point (η, ζ) is proportional to η. Assuming that the Hooke's law is satisfied, the total moment of forces stretching and compressing the wood fibres may be computed by integrating the moment of a linear function $\sigma_\mathrm{b}(\eta, \zeta)$

$$\begin{aligned} M(x) &= \int_{K(x)} \eta\sigma_\mathrm{b}(\eta,\zeta)\,\mathrm{d}\eta\,\mathrm{d}\zeta = \int_{K(x)} \frac{\sigma_{\mathrm{b\,max}}}{r(x)}\eta^2\,\mathrm{d}\eta\,\mathrm{d}\zeta \\ &= \frac{\sigma_{\mathrm{b\,max}}}{r(x)} \int_0^{r(x)} \left(\int_0^{2\pi} t^3 \cos^2\theta\,\mathrm{d}\theta \right) \mathrm{d}t = \frac{\pi}{4} r^3(x)\sigma_{\mathrm{b\,max}}. \end{aligned}$$

The symbol $\sigma_{\mathrm{b\,max}}$ denotes the maximal compressing tension caused by bending in the section area. The torque calculated above balances the sum of bending torques caused by the wind and the gravitation.

With the assumption that the maximal wind force per unit length, F, is constant along the trunk, the bending torque caused by the wind is

$$M_\mathrm{w}(x) = \int_x^H F(t-x)\,\mathrm{d}t = \frac{1}{2}F(H-x)^2.$$

The assumption about the distribution of weight of branches makes it possible to calculate the bending torque caused by gravitation

$$M_\mathrm{g}(x) = D\tan\alpha \int_x^H r^2(t)(t-x)\,\mathrm{d}t.$$

We assume that α is small, so that $\sin\alpha \approx \alpha \approx \tan\alpha$.

THE MODEL

Trunk shape equation

The maximal stretching tension in the section $K(x)$ is $\sigma_{\mathrm{c}}(x) - \sigma_{\mathrm{b\,max}}(x)$, while the maximal compressing tension is $\sigma_{\mathrm{c}}(x) + \sigma_{\mathrm{b\,max}}(x)$; due to the orientation of the gravitation force, the absolute value of the latter is greater. Therefore we assume that the durability of the trunk is determined by the limit of the compressing tension for the wood, which we denote by $\sigma_{\max}$. The basic idea of the model is, that during the strongest possible wind the maximal compressing tension for all x is equal to this limit. It is described by the following equation

$$\begin{aligned}\sigma_{\max} &= \sigma_{\mathrm{b\,max}}(x) + \sigma_{\mathrm{c}}(x) = \sigma_{\mathrm{w}}(x) + \sigma_{\mathrm{g}}(x) + \sigma_{\mathrm{c}}(x)\\ &= \frac{2F(H-x)^2}{\pi r^3(x)} + \frac{4D\tan\alpha}{\pi r^3(x)}\int_x^H r^2(t)(t-x)\,\mathrm{d}t + \frac{D}{\pi r^2(x)}\int_x^H r^2(t)\,\mathrm{d}t.\end{aligned}$$

The functions σ_{w} and σ_{g} describe respectively the maximal compressing tensions caused by the wind and bending due to the trunk slant. It is convenient to replace the parameters F, D and $\sigma_{\max}$ by $f = F/\sigma_{\max}$ and $d = D/\sigma_{\max}$. Let

$$\begin{aligned} n_{\mathrm{w}}(x) &= \frac{\sigma_{\mathrm{w}}(x)}{\sigma_{\max}} = \frac{2f(H-x)^2}{\pi r^3(x)},\\ n_{\mathrm{g}}(x) &= \frac{\sigma_{\mathrm{g}}(x)}{\sigma_{\max}} = \frac{4d\tan\alpha}{\pi r^3(x)}\int_x^H r^2(t)(t-x)\,\mathrm{d}t,\\ n_{\mathrm{c}}(x) &= \frac{\sigma_{\mathrm{s}}(x)}{\sigma_{\max}} = \frac{d}{\pi r^2(x)}\int_x^H r^2(t)\,\mathrm{d}t.\end{aligned}$$

The functions defined above describe respectively shares of tensions caused by the wind, bending due to the trunk slant and compressing by the weight in the total maximal compressing tension at the level x. They are nonnegative and

$$n_{\mathrm{w}}(x) + n_{\mathrm{g}}(x) + n_{\mathrm{c}}(x) = 1 \quad \text{for } x \in [0, H). \tag{1}$$

Remark. This equation may be rewritten in the form

$$\pi r^3(x) - d\int_x^H \big(4\tan\alpha(t-x) + r(x)\big) r^2(t)\,\mathrm{d}t = 2f(H-x)^2.$$

A solution to be found must satisfy the boundary conditions $r(0) = r_0$ and $r(H) = 0$, where r_0 (radius of the section at the ground level) and H (trunk height) are obtained by measurement. The parameter α, as explained in Introduction, is the maximal slant for the species. The parameters α, f and d may be chosen in experiments, so as to obtain solutions which determine shapes possibly close to the shapes of trunks of living trees. The value of f is related with the dimensions H and r_0 and by the share $n_{\mathrm{w}0} = n_{\mathrm{w}}(0)$ of the tension caused by wind at the ground level. In experiments one can take arbitrarily $n_{\mathrm{w}0} \in (0, 1)$ and compute

$$f = \frac{\pi r_0 n_{\mathrm{w}0}}{2H^2}.$$

The value of d must be chosen so as to obtain a function $r(x)$ satisfying (1) *and* the boundary conditions; for the discrete model it is done by solving a system of nonlinear equations.

Discrete model and numerical algorithm

The numerical solution of (1) may be found using the collocation method [2] as follows. The interval $[0, H]$ is divided to N parts, each of length $h = H/N$. The approximate solution, $r^{(h)}(x)$, is assumed to be a continuous spline of degree 1, whose knots are the numbers ih, where $i = 0, \ldots, N$. The collocation at these knots yields the system of equations

$$n_{\mathrm{w}}^{(h)}(ih) + n_{\mathrm{g}}^{(h)}(ih) + n_{\mathrm{c}}^{(h)}(ih) - 1 = 0 \quad \text{for } i = 0, \ldots, N-1. \tag{2}$$

Given f, α and the boundary conditions $r^{(h)}(0) = r_0 > 0$ and $r^{(h)}(H) = r_N = 0$, the unknown quantities are the function values $r_i = r^{(h)}(ih)$ for $i = 1, \dots, N-1$ and the parameter d. The number of unknowns is thus equal to the number of equations.

With the assumed form of the numerical solution, the integrands in the expressions for shares $n_{\mathrm{g}}^{(h)}(x)$ and $n_{\mathrm{c}}^{(h)}(x)$ are quadratic or cubic polynomials in each interval $[ih, (i+1)h]$. Therefore the integrals may be computed exactly using the Simpson's quadrature [2]. Using it we obtain

$$\begin{aligned} a_j &\stackrel{\text{def}}{=} \int_{jh}^{(j+1)h} \big(r^{(h)}(t)\big)^2 \,\mathrm{d}t = \frac{h}{3}(r_j^2 + r_j r_{j+1} + r_{j+1}^2), \\ b_{ij} &\stackrel{\text{def}}{=} \int_{jh}^{(j+1)h} \big(r^{(h)}(t)\big)^2 (t - ih)\,\mathrm{d}t \\ &= \frac{h^2}{3}\big((j-i+1/4)r_j^2 + (j-i+1/2)r_j r_{j+1} + (j-i+3/4)r_{j+1}^2\big). \end{aligned}$$

Let

$$A_i \stackrel{\text{def}}{=} \int_{ih}^{H} \big(r^{(h)}(t)\big)^2 \,\mathrm{d}t = \sum_{j=i}^{N-1} a_j, \qquad B_i \stackrel{\text{def}}{=} \int_{ih}^{H} \big(r^{(h)}(t)\big)^2 (t-ih)\,\mathrm{d}t = \sum_{j=i}^{N-1} b_{ij}.$$

Using these symbols, the system (2) may be rewritten in the form

$$g_i(r_1, \dots, r_{N-1}, d) = 0, \quad i = 0, \dots, N-1, \tag{3}$$

where

$$g_i(r_1, \dots, r_{N-1}, d) \stackrel{\text{def}}{=} \frac{2f\big((N-i)h\big)^2}{\pi r_i^3} + \frac{4d\tan\alpha}{\pi r_i^3} B_i + \frac{d}{\pi r_i^2} A_i - 1.$$

The system (3) may be solved using the Newton's method. The necessary derivatives of the functions g_i are

$$\begin{aligned} \frac{\partial g_i}{\partial r_k} &= 0 \quad \text{for } k = 0, \dots, i-1, \\ \frac{\partial g_i}{\partial r_i} &= \frac{-6f\big((N-i)h\big)^2}{\pi r_i^4} + \frac{4d\tan\alpha}{\pi r_i^3}\left(\frac{\partial B_i}{\partial r_i} - \frac{3}{r_i}B_i\right) + \frac{d}{\pi r_i^2}\left(\frac{\partial A_i}{\partial r_i} - \frac{2}{r_i}A_i\right), \\ \frac{\partial g_i}{\partial r_k} &= \frac{4d\tan\alpha}{\pi r_i^3}\frac{\partial B_i}{\partial r_k} + \frac{d}{\pi r_i^2}\frac{\partial A_i}{\partial r_k} \quad \text{for } k = i+1, \dots, N-1, \\ \frac{\partial g_i}{\partial d} &= \frac{1}{\pi r_i^2}\left(\frac{4\tan\alpha}{r_i}B_i + A_i\right). \end{aligned}$$

To the formulae above we substitute

$$\frac{\partial A_i}{\partial r_i} = \frac{\partial a_i}{\partial r_i} = \frac{h}{3}(2r_i + r_{i+1}), \qquad \frac{\partial B_i}{\partial r_i} = \frac{\partial b_{ii}}{\partial r_i} = \frac{h^2}{6}(r_i + r_{i+1}),$$

and for $k > i$

$$\begin{aligned} \frac{\partial A_i}{\partial r_k} &= \frac{\partial a_{k-1}}{\partial r_k} + \frac{\partial a_k}{\partial r_k} = \frac{h}{3}(r_{k-1} + 4r_k + r_{k+1}), \\ \frac{\partial B_i}{\partial r_k} &= \frac{\partial b_{i,k-1}}{\partial r_k} + \frac{\partial b_{ik}}{\partial r_k} = \frac{h^2}{3}\big((k-i-1/2)r_{k-1} + 4(k-i)r_k + (k-i+1/2)r_{k+1}\big). \end{aligned}$$

The derivative matrix of the vector function $\boldsymbol{g}$ made of $g_0, \dots, g_{N-1}$ is upper Hessenberg [2]. Its coefficients may be computed efficiently, using the formulae $A_N = B_N = 0$ and

$$\begin{aligned} A_i &= a_i + A_{i+1}, \\ B_i &= b_{ii} + B_{i+1} + hA_{i+1} \quad \text{for } i = 0, \dots, N-1. \end{aligned}$$

The initial point for the Newton's method (good enough for all experiments made by the author) may consist of the numbers $r_i^{(0)} = r_0(N-i)/N$ (these numbers describe a cone) and the number

$$d^{(0)} = 3\frac{\pi r_0^3 - 2fH^2}{Hr_0^2(H\tan\alpha + r_0)},$$

obtained by solving the equation $g_0(r_1^{(0)},\dots,r_{N-1}^{(0)},d) = 0$.

EXPERIMENTS AND CONCLUSIONS

The dimensions of the Giant Sequoia in Jasov were measured in August 2012 [3]; the height of the tree was $47.7\mathrm{m}$ and the girth of its trunk, $1.3\mathrm{m}$ above the ground level, was $7.42\mathrm{m}$. The parameters taken for the numerical experiments were $H = 47.7$, $r_0 = 1.1$ and $N = 1024$.

Solutions were found for four values of $n_{\mathrm{w}0}$ and four angles α. Some results of the 16 experiments are gathered in Table 1. Its last three columns show the shares of the tensions at the ground level. Figure 1 shows the graphs of the solutions; to improve the visibility of the influence of the parameters on the solution, different scales were assumed for each axis.

Table 1. Parameters of numerical solutions and shares of tensions at the ground level

α	f [m]	d [m^{-1}]	$n_{\mathrm{w}0}$	$n_{\mathrm{g}0}$	$n_{\mathrm{c}0}$
$0°$	$4.594\cdot10^{-5}$	0.2731	0.05	0	0.95
$0°$	$9.189\cdot10^{-5}$	0.2263	0.1	0	0.9
$0°$	$1.838\cdot10^{-4}$	0.1740	0.2	0	0.8
$0°$	$3.676\cdot10^{-4}$	0.1121	0.4	0	0.6
$0.5°$	$4.594\cdot10^{-5}$	0.1942	0.05	0.2308	0.7192
$0.5°$	$9.189\cdot10^{-5}$	0.1620	0.1	0.2318	0.6682
$0.5°$	$1.838\cdot10^{-4}$	0.1246	0.2	0.2187	0.5813
$0.5°$	$3.676\cdot10^{-4}$	0.07954	0.4	0.1740	0.4260
$1°$	$4.594\cdot10^{-5}$	0.1515	0.05	0.3737	0.5763
$1°$	$9.189\cdot10^{-5}$	0.1265	0.1	0.3697	0.5303
$1°$	$1.838\cdot10^{-4}$	0.09711	0.2	0.3437	0.4563
$1°$	$3.676\cdot10^{-4}$	0.06165	0.4	0.2697	0.3303
$2°$	$4.594\cdot10^{-5}$	0.1054	0.05	0.5388	0.4112
$2°$	$9.189\cdot10^{-5}$	0.08804	0.1	0.5252	0.3748
$2°$	$1.838\cdot10^{-4}$	0.06742	0.2	0.4811	0.3189
$2°$	$3.676\cdot10^{-4}$	0.04253	0.4	0.3721	0.2279

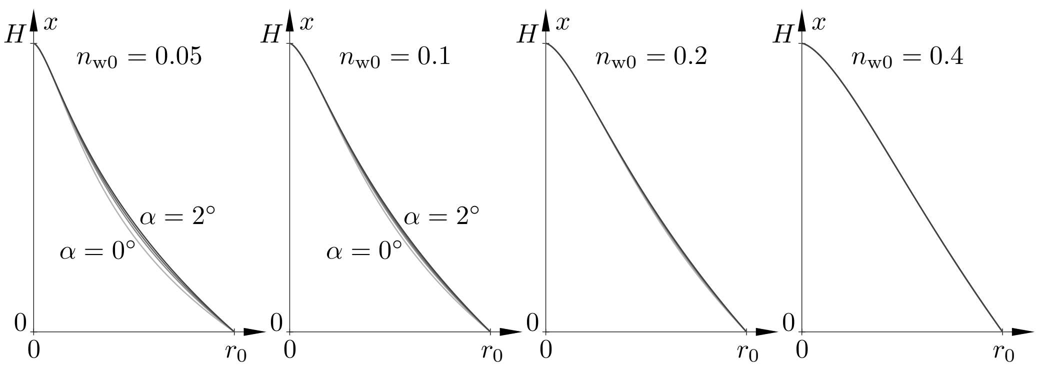

Figure 1. Graphs of numerical solutions.

As one can see, the influence of the parameter α on the trunk shape decreases with growth of the reserve of durability for strong winds.

Figure 2 shows graphs of the tensions σ_{w}, σ_{g} and σ_{c} for a trunk, whose shape corresponds to the parameters $\alpha = 1°$ and $n_{\mathrm{w}0} = 0.1$. The graphs on the left side show the case of $\beta = \alpha$. On the right side one can see what happens when the trunk inclination angle is smaller than α.

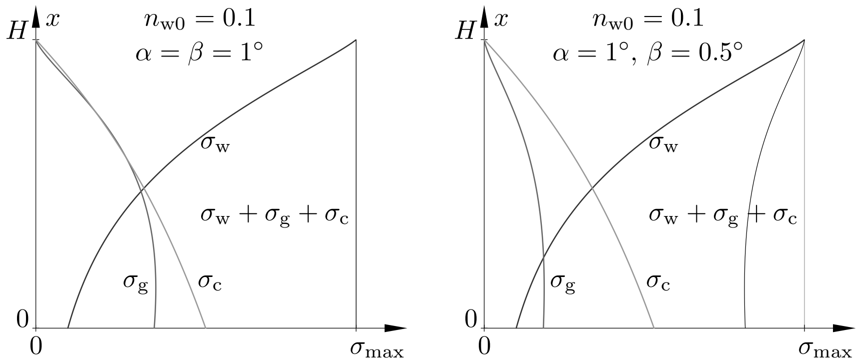

Figure 2. Maximal compressing tensions along the trunk.

Figure 3 shows the tensions $\sigma_{\mathrm{w}}(x)$, $\sigma_{\mathrm{g}}(x)$ and $\sigma_{\mathrm{s}}(x)$ and their sums when the wind force exceeds the limit by 10% and by 20%. As one can see, it is most likely that the trunk breaks close to the top, which ensures some protection to the lower part of the tree. This observation, in the author's opinion, provides a justification for the model.

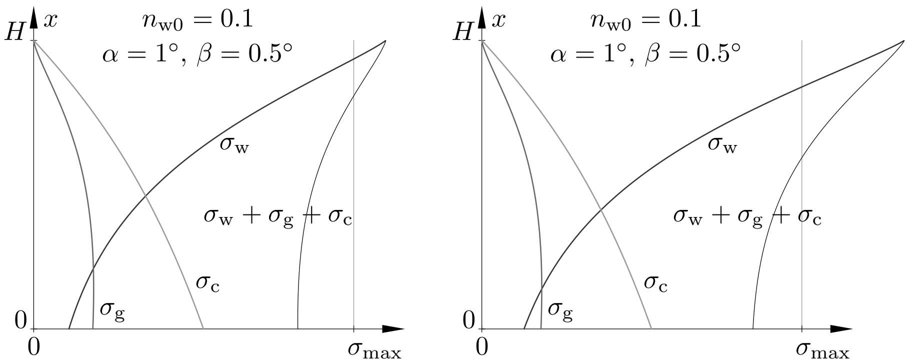

Figure 3. Maximal compressing tensions for the wind too strong.

In the experiments described above the parameters of the model were chosen so as to obtain solutions which look like the tree from Jasov on photographs. The parameters might be chosen more precisely if measurements of the trunk diameter at a number of levels were available.

The model developed in this paper makes it possible to obtain quite a lot of information about the tensions based only on some measurements of the trunk geometry—without measuring the actual forces nor mechanical properties of wood. A simpler model described in [4] does not take into account the slant of the trunk. Although it is also possible to choose the parameters of that model so as to approximate the trunk shape, the tensions obtained using the model described here seem much closer to reality.

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