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NUMERICAL METHODS FOR DELAY DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO EPIDEMIC MODELS

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ABSTRACT

In the present talk, we introduce a numerical method based on operator splitting techniques and Magnus integrators for solving delay differential equations. To perform a rigorous error analysis, we employ tools from functional analysis, such as operator semigroups and evolution families.

Our motivating example is the classical SIR epidemic model with a latent period $\delta > 0$, supplemented by vaccination and random movement of individuals over a domain $\Omega \subset \mathbb{R}^2$. Writing the proportions of the susceptible, infected, and recovered populations at time $t > 0$ and position $\mathbf{x} = (x, y) \in \Omega$ as $S(t, \mathbf{x})$, $I(t, \mathbf{x})$, and $R(t, \mathbf{x})$, respectively, the model has the form

$$\begin{cases} \partial_t S(t, \mathbf{x}) = \partial_x(\kappa(\mathbf{x})\partial_x S(t, \mathbf{x})) + \partial_y(\kappa(\mathbf{x})\partial_y S(t, \mathbf{x})) - \beta S(t, \mathbf{x})I(t - \delta, \mathbf{x}) - \nu S(t, \mathbf{x}), \\ \partial_t I(t, \mathbf{x}) = \partial_x(\kappa(\mathbf{x})\partial_x I(t, \mathbf{x})) + \partial_y(\kappa(\mathbf{x})\partial_y I(t, \mathbf{x})) + \beta S(t, \mathbf{x})I(t - \delta, \mathbf{x}) - \gamma I(t, \mathbf{x}), \\ \partial_t R(t, \mathbf{x}) = \partial_x(\kappa(\mathbf{x})\partial_x R(t, \mathbf{x})) + \partial_y(\kappa(\mathbf{x})\partial_y R(t, \mathbf{x})) + \nu S(t, \mathbf{x}) + \gamma I(t, \mathbf{x}), \end{cases} \quad (1)$$

with $\beta, \gamma, \nu > 0$ representing the infection, recovery, and vaccination rates, respectively, and $\kappa(\mathbf{x}) > 0$ denoting the diffusion coefficient. We impose homogeneous Neumann boundary condition, that is, the normal derivatives of S , I , and R vanish on the boundary of Ω , meaning that individuals cannot leave the domain.

The system (1) can be written as an abstract quasilinear delay differential equation on the space $X = C(\Omega)^3$ for the function $u(t) = (S(t, \cdot), I(t, \cdot), R(t, \cdot))$ in the form

$$\begin{cases} u'(t) = (A + Q(u(t - \delta)))u(t), & t \in [0, \infty), \\ u(\sigma) = \varphi(\sigma), & \sigma \in [-\delta, 0] \end{cases} \quad (2)$$

with a given initial history function $\varphi: [-\delta, 0] \rightarrow X$, an appropriate unbounded operator $A: D(A) \subset X \rightarrow X$, and a sufficiently smooth function Q taking values in the space of bounded linear operators on X .

In our presentation, we consider a wide class of semilinear or quasilinear delay differential equations of the general form (2) and present a fully discrete time-and-space numerical method in an abstract way. To this end, we apply an operator splitting technique to decompose the problem

into simpler subproblems such as

$$\begin{aligned} u_1'(t) &= Au_1(t), \\ u_2'(t) &= B(t)u_2(t) \quad \text{with} \quad B(t) := Q(u_2(t - \delta)). \end{aligned}$$

The solution of the first subproblem is given in terms of an operator semigroup, which can be regarded as the exponential of the (unbounded) operator A in an appropriate sense. Since the second subproblem is nonautonomous, its solution is given by an evolution family, which can also be approximated by a semigroup using Magnus's idea. With a timestep $\tau > 0$, this results in the time discretization

$$u((n+1)\tau) \approx e^{\frac{\tau}{2}B((n+1/2)\tau)} e^{\tau A} e^{\frac{\tau}{2}B((n+1/2)\tau)}, \quad n \in \mathbb{N},$$

where the operator semigroup operators (denoted by the exponentials) need to be further approximated by a dimension reduction—typically a spatial discretization (finite differences or finite element method), but also allowing for a Fourier or Galerkin approach.

The aim of the talk is to show that operator semigroups provide an efficient tool for designing and analysing numerical methods even for nonlinear problems such as abstract semilinear or quasilinear delay equations, with a particularly relevant application to epidemic modeling. To this end, we will briefly introduce the necessary notions from operator theory and numerical analysis. We then outline the main ideas of the error analysis, which show that the method preserves the orders of the time and space discretization errors. This, in turn, means that the total error separates into a second-order time discretization error (arising from the Strang splitting and the Magnus method) and an independent space discretization error. Consequently, matching the space discretization to the timestep in order to obtain an optimal convergence rate becomes straightforward. We illustrate our results on the delay SIR epidemic model (1) and present numerical simulations as well.

The talk is based on our previous works. In [1], the Magnus method was introduced for ordinary delay differential equations. We showed its second-order convergence in time and its positivity preserving property. The work [2] considered the same time discretization method for partial delay differential equations, that is, for infinite dimensional spaces. We showed the existence of an exact solution, the second-order convergence of the method, and the preservation of invariant sets. As an application, we considered the model (1) and showed that the method preserves the positivity of the populations S, I, R and the size $S + I + R$ of the total population. In our most recent study [3], we generalize our results to allow for (almost) arbitrary space discretizations, which leads to a method that also preserves the order of the space discretization error. All of our papers include numerical experiments illustrating our theoretical results.

REFERENCES

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