Krajowa Konferencja Zastosowań Matematyki w Biologii i Medycynie

Sandomierz, 5th–9th September 2016

SEARCHING FOR INSTABILITY

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ABSTRACT

In this paper the conditions for the occurrence of solution instability in systems of PDE's is discussed. As the examples we take several systems of PDE's occurring in mathematical biology describing the species reacting by chemotaxis, chemorepulsion and aerotaxis respectively. The method which we attempt to present is not yet fully develop and understood.

INTRODUCTION

In many cases one has to deal with systems of equations containing large or small parameters. Usually this permits one to reduce the system by neglecting some terms. The linear hyperbolic operators, which by this sort of procedure can be reduced to hyperbolic operators of lower order, were considered by G. B. Whitham [1]. Roughly speaking, he noticed that if it happens that some of characteristic velocities of the reduced system contradict the causality of the original system then one encounters the instability of solutions of the original system. As a simple example let us take the following linear system [2]

$$\begin{cases} u_t + u_x + v_x = 0, \\ v_t + 2v_x = \mu(ru - v). \end{cases} \tag{1}$$

For large μ we may take $v = ru$ which leads to the reduced system containing only one equation

$$u_t + (1 + r)u_x = 0. \tag{2}$$

The characteristic speeds of the original system are $c_1 = 1$ and $c_2 = 2$, whereas for the reduced one it is equal to $c_0 = 1 + r$. Clearly, the wave type solution of the equation (2) traveling with the speed $c_0 = 1 + r$ is an approximate solution of (up to terms of order $\frac{1}{\mu}$) of the original system (1) if $1 \leqslant 1 + r \leqslant 2$; otherwise the causality relation for the system (1) would be violated. As one can easily demonstrate the "violation of causality" of the reduced system with respect to system (1) implies the instability of some solutions of the original system. As one can expect, the argument of causality violation is also applicable in the case of nonlinear hyperbolic systems of equations.
In this paper we try to extend this approach to parabolic systems describing the reacting biological species. More precisely, we will analyze the systems of:

1. chemotaxis - the movement of an organism in the direction of increasing concentration of chemical stimulus,
2. competing species with chemorepulsion,
3. symbiotic species with aerotaxis.

THE METHOD

Let us consider the following example

$$\begin{cases} u_t = \Delta u - K\Delta v, \\ v_t = \Delta v - av + bu, \end{cases} \tag{3}$$

where K, a and b are positive constants.

We will see that the initial value problem for the system (3) is well posed. When the constants a and b are large one is tempted to neglect the other terms in the second equation to obtain

$$v = \frac{b}{a}u.$$

Inserting this relation to the first equation of the system (3) we arrive at the "reduced system"

$$u_t = (1 - \frac{b}{a}K)\Delta u. \tag{4}$$

However the last equation becomes "bad" when $1 - \frac{b}{a}K$ is negative. Thus the initial value problem for the equation (4) is ill posed for $\frac{b}{a}K > 1$ and we observe the Hadamard instability. We will demonstrate that the appearance of Hadamard instability of Eq.(4) corresponds to the instability of solutions of the system (3).

Taking the Fourier transform of the system (3) we obtain the ODE system

$$\begin{cases} \hat{u}_t = -\xi^2\hat{u} + K\xi^2\hat{v}, \\ \hat{v}_t = -\xi^2\hat{v} - a\hat{v} + b\hat{u}. \end{cases} \tag{5}$$

The system (5) can be written in the form

$$\begin{bmatrix} \hat{u}_t \\ \hat{v}_t \end{bmatrix} = \begin{bmatrix} -\xi^2 & K\xi^2 \\ b & -a-\xi^2 \end{bmatrix} \cdot \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}. \tag{6}$$

The eigenvalues of the matrix of the system (6) are

$$\lambda_+ = \sqrt{\left(\frac{a}{2}\right)^2 + bK\xi^2} - (\xi^2 + \frac{a}{2}), \quad \lambda_- = -\sqrt{\left(\frac{a}{2}\right)^2 + bK\xi^2} - (\xi^2 + \frac{a}{2})$$

and the corresponding eigenvectors are

$$Y_+ = \begin{bmatrix} K\xi^2 \\ \xi^2 + \lambda_+ \end{bmatrix}, \quad Y_- = \begin{bmatrix} K\xi^2 \\ \xi^2 + \lambda_- \end{bmatrix}.$$

The Fourier transform of the solution of Eqs. (6) has the form

$$\begin{bmatrix} \hat{u}(t,\xi) \\ \hat{v}(t,\xi) \end{bmatrix} = A(\xi)Y_+e^{\lambda_+ t} + B(\xi)Y_-e^{\lambda_- t}, \tag{7}$$

where $A(\xi)$ and $B(\xi)$ are determined by Fourier transform of the initial data $\hat{u}_0(\xi)$ and $\hat{v}_0(\xi)$. Finally we get

$$\begin{cases} \hat{u}(t,\xi) = \left(\frac{\hat{u}_0}{2} + \frac{2\hat{v}_0K\xi^2 + \hat{u}_0a}{2(\lambda_+ - \lambda_-)}\right) \cdot e^{\lambda_+ t} + \left(\frac{\hat{u}_0}{2} - \frac{2\hat{v}_0K\xi^2 + \hat{u}_0a}{2(\lambda_+ - \lambda_-)}\right) \cdot e^{\lambda_- t}, \\ \hat{v}(t,\xi) = \left(\frac{\hat{v}_0}{2} + \frac{2\hat{u}_0b - \hat{v}_0a}{2(\lambda_+ - \lambda_-)}\right) \cdot e^{\lambda_+ t} + \left(\frac{\hat{v}_0}{2} - \frac{2\hat{u}_0b - \hat{v}_0a}{2(\lambda_+ - \lambda_-)}\right) \cdot e^{\lambda_- t}, \end{cases} \tag{8}$$

where $\lambda_+ - \lambda_- = 2\sqrt{(\frac{a}{2})^2 + bK\xi^2}$ does not vanish for $a > 0$.

Existance and regularity

One can easily notice that if the initial functions $\hat{u}_0$ and $\hat{v}_0$ are from $L^2(\mathbb{R}^n)$ then the Fourier transform (8) is bounded and vanishing for $\xi \to \infty$ faster then $e^{-p\xi^2}$ for some $p > 0$. Thus $\hat{u}(t,\xi)$ and $\hat{v}(t,\xi)$ for constant $t > 0$ are square integrable with any power of ξ. Therefore $u(t,x)$ and $v(t,x)$, for $t > 0$, are smooth functions belonging to Sobolev space H^s for any $s > 0$. (Recall that $f \in H^s \iff \int\limits_{\mathbb{R}^n} |\hat{f}(\xi)|^2(1+|\xi|^2)^s d\xi < \infty$ for $f : \mathbb{R}^n \to \mathbb{R}$.) Finally we can conclude that $(u(t,x), v(t,x)) \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$.

Instability

Let us notice that λ_+ can be written as

$$\lambda_+ = \frac{(\frac{a}{2})^2 + bK\xi^2 - \xi^4 - \xi^2 a - (\frac{a}{2})^2}{\sqrt{(\frac{a}{2})^2 + bK\xi^2} + (\xi^2 + \frac{a}{2})} = \xi^2 \frac{(bK-a) - \xi^2}{\sqrt{(\frac{a}{2})^2 + bK\xi^2} + (\xi^2 + \frac{a}{2})}.$$

Therefore λ_+ becomes positive if $bK - a > 0$ and $\xi^2 < bK - a$. Thus we observe the instability of modes satisfying $\xi^2 < bK - a$, or more precisely, for any ξ_0 such that $\xi_0^2 \in (0, bK-a)$ the expression

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} K\xi_0^2 \\ \xi_0^2 + \lambda_+(\xi_0) \end{bmatrix} e^{\lambda_+(\xi_0)t - i\xi_0 x}$$

is a solution of Eqs. (3) growing exponentially in time. The fastest growing modes are those for which λ_+ is close to its maximal value. For $(bK-a)$ small and positive this is $\xi^2 \approx \frac{1}{2}(bK-a)$. Thus indeed, the condition for Hadamard instability of the reduced system implies the instability of solutions of the original system (3).

EXAMPLES FROM BIOLOGY

Chemotaxis

Chemotaxis is a response elicited by chemicals: that is, a response to a chemical concentration gradient.[3] The pioneering works of Patlak in 1953 [4] and Keller and Segel in 1970-1971 [5, 6] give the system

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t = \Delta c - \alpha c + \beta n, \end{cases} \tag{9}$$

where $n = n(t,x)$ is the cell density and $c = c(t,x)$ is the concentration of chemical attractant substance. Also, χ is the chemotatic sensitivity, $\alpha \geqslant 0$ is the decay rate of the chemical and $\beta \geqslant 0$ is its production rate.[7]

Neglecting the terms with derivatives in the second equation of the system (9) we have

$$c = \frac{\beta}{\alpha} n.$$

Inserting the above expressions into the first equation of the system (9) we arrive at the "reduced system"

$$m_t = \Delta m - div \cdot \nabla\gamma(m), \tag{10}$$

where $m = \frac{\beta}{\alpha} n$ and $\gamma(m)$ is the primitive function of $m\chi(m)$. The initial value problem for the Eq. (10) is ill posed if $\gamma'(m) > 1$. Indeed, developing Eq. (10) we get

$$m_t = (1 - \gamma'(m))\Delta m - \gamma''(m)(\nabla m)^2.$$

This suggests that one encounters instability of solutions of the system (9) if $1 - \gamma'(m) < 0$. Linearization of the system (9) around its constant states leads to the system which basically is identical with system (3). Therefore, the constant equilibrium solutions are unstable if $\gamma'(m) > 1$. However one can expect that also the solutions which vary with time and space may be unstable if this condition is satisfied in a certain region for $t > 0$.

Competing species with chemorepulsion

Chemorepulsion is an example of a negative taxis, since in this case organisms are moving away from the substance. Consider the model of interspecies competition with chemorepulsion [8]

$$\begin{cases} n_t = D_1\Delta n + \nabla\cdot(n\chi\nabla c) + \mu n(1-n-ap), \\ p_t = D_2\Delta p + \nu p(1-p-bn), \\ c_t = D_3\Delta c - \rho c + \sigma p, \end{cases} \tag{11}$$

where $n = n(t,x)$ and $p = p(t,x)$ are densities of competing species and $c = c(t,x)$ is the chemical (signal) produced by the second specie. Moreover, $\chi > 0$ is the chemorepulsion coefficient, $\mu > 0$ and $\nu > 0$ are the population growth rates, $a > 0$ and $b > 0$ describe the strength of competition, while ρ and σ describe the rates of signal production and signal degradation respectively. We also assume that the diffusion coefficients D_1, D_2 and D_3 are positive.
Note that for positive chemotaxis the sign standing at the term with χ is negative, whereas it is positive in case of chemorepulsion. For simplicity we consider Eqs. (11) in the whole space ($x \in \mathbb{R}^n$). The case of bounded domain is considered in [8]. Similarly as in [8] we will confine ourself to the case of weak competition, ie. $0 < a < 1$ and $0 < b < 1$.
In the absence of the diffusion, the steady state

$$(n^*, c^*, p^*) = \Big(\frac{1-a}{1-ab}, \frac{\sigma}{\rho}\frac{1-b}{1-ab}, \frac{1-b}{1-ab}\Big)$$

of the system (11) is stable if the matrix of the system is negative definite. In our case the matrix is given by

$$\begin{bmatrix} -\mu & -\mu a & 0 \\ -\nu b & -\nu & 0 \\ 0 & \sigma & -\rho \end{bmatrix} \tag{12}$$

Under our assumption ($\mu > 0$, $\nu > 0$, $\rho > 0$ and $0 < ab < 1$) the matrix (12) is negative definite. Standard linearization of the system (11) at the steady state (n^*, c^*, p^*) leads to the following system

$$\begin{cases} n_t = D_1\Delta n + \chi n^*\Delta c + \mu n - 2\mu n^* n - \mu a n^* p - \mu a p^* n, \\ p_t = D_2\Delta p + \nu p - 2\nu p^* p - \nu b p^* n - \nu b n^* p, \\ c_t = D_3\Delta c - \rho c + \sigma p. \end{cases} \tag{13}$$

Taking the Fourier transform of Eqs. (13) with respect to the space variables we arrive at the system of ODE's. The corresponding matrix of this system has the form

$$A(\xi) = \begin{bmatrix} -D_1\xi^2 - \mu u^* & -\mu a u^* & -\chi\xi^2 u^* \\ -\nu b v^* & -D_2\xi^2 - \nu v^* & 0 \\ 0 & \sigma & -D_3\xi^2 - \rho \end{bmatrix}.$$

Note that

$$\det A(\xi) = -(D_1\xi^2 + \mu u^*)(D_2\xi^2 + \nu v^*)(D_3\xi^2 + \rho) + \nu b\sigma\chi\xi^2 u^* v^* + \mu\nu a b u^* v^*(D_3\xi^2 + \rho) > 0.$$

Thus the linear stability analysis leads to the following condition for instability of the constant steady state

$$\chi > \min_\xi \frac{(D_1\xi^2 + \mu u^*)(D_2\xi^2 + \nu v^*)(D_3\xi^2 + \rho) - \mu\nu a b u^* v^*(D_3\xi^2 + \rho)}{\nu b\sigma\xi^2 u^* v^*}$$

It means that in the case of weak competition the constant steady state may be destabilized if χ is large enough.
Let us note that for small μ, large ν and ρ the condition of instability can be approximated by

$$\chi > \frac{D_1\rho}{\sigma b n^*}. \tag{14}$$

Assuming large ρ and ν we have the approximate relations:

$$c = \frac{\sigma}{\rho}, \quad p = 1 - bn, \quad c = \frac{\sigma}{\rho}(1 - bn).$$

Applying them for to the first equation of the system (11) we arrive at the "wrong" equation for n

$$n_t = (D_1 - \chi \frac{\sigma b}{\rho} n)\Delta n + \dots,$$

if only $D_1 - \chi \frac{\sigma b}{\rho} n < 0$. Note that the last (Hadamard instability) condition is the same as (14). This again shows that there is connection between the ill posedness of the reduced system and the instability of solutions of the original system of Eqs. (11) for some range of parameters.

Aerotaxis

Aerotaxis is the response of an organism to variation in oxygen concentration, and is mainly found in aerobic bacteria.[3] In aerotaxis, oxygen dissolved in water plays the role of both attractant (at moderate concentrations) and repellent (at high and low concentrations).[9]

Let us consider now two species: aerobic bacteria consuming oxygen and producing carbon dioxide along with phytoplankton producing oxygen and consuming carbon dioxide. This symbiotic interaction can be modeled by the following system

$$\begin{cases} n_t = D_1 \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t = D_2 \Delta c - \sigma c - k(c)n + rp, \\ p_t = D_3 \Delta p + p(K - \beta p + \alpha n), \end{cases} \tag{15}$$

where $n = n(t,x)$ is the cell density, $c = c(t,x)$ is the oxygen concentration and $p = p(t,x)$ is the plankton density. Moreover, $\chi(c)$ is the aerobic sensitivity and $k(c)$ is the oxygen consumption rate, wherein $k(c) = k$ for $c \geqslant 0$ and $k(c) = 0$ for $c < 0$. Also, $\sigma \geqslant 0$ is the decay rate of the oxygen (as it released to the atmosphere), $r \geqslant 0$ is its production rate by phytoplankton, K is the capacity of the environment, $\alpha \geqslant 0$ is the coefficient of symbiosis related to the production of the carbon dioxide by aerobic bacteria, $\beta \geqslant 0$ is responsible for the reduction of the multiplication rate of plankton due to its increasing density. We also assume that the diffusion coefficients D_1, D_2 and D_3 are positive.

Omitting the terms with derivatives in the last equation of the system (15) we have

$$p = \frac{K + \alpha n}{\beta}.$$

When we interpose this relation into the second equation of the system (15) we get

$$c_t = D_2 \Delta c - \sigma c - kn + r\frac{K + \alpha n}{\beta}. \tag{16}$$

Similarly, neglecting the terms with derivatives in the equation (16) we have

$$c = \Big(\frac{r\alpha}{\beta\sigma} - \frac{k}{\sigma}\Big)n + \frac{rK}{\beta\sigma}.$$

Inserting the above expressions into the first equation of the system (15) we arrive at

$$m_t = D_1 \Delta m - div \cdot \nabla \gamma(m), \tag{17}$$

where $m = (\frac{r\alpha}{\beta\sigma} - \frac{k}{\sigma})n + \frac{rK}{\beta\sigma}$ and $\gamma(m)$ is the primitive function of $m\chi(m)$. Transforming the equation (17) we obtain

$$m_t = (D_1 - \gamma'(m))\Delta m - \gamma''(m)(\nabla m)^2.$$

Obviously, the initial value problem for Eq. (16) is ill posed for $\gamma'(m) > D_1$. This again suggests that if $D_1 - \gamma'(m) < 0$ then one should expect the instability of solutions of the system (15) for some range of parameters.

CONCLUDING REMARKS

The problem of stability of solutions is very important in the applied sciences. There are various methods of investigation whether the solution is stable or not, e.g. by linearization or by studying the energy of the system. Typically in every nontrivial case a piece of hard work must be done. Therefore finding new and easy instability criteria may be very useful. In this paper three examples of systems of PDE's are analyzed. By neglecting some terms we obtained so called reduced systems. We tried to demonstrate that if the initial value problem for the reduced system is ill posed then one can have instability of solutions of the original system for the similar range of parameters. Our study should be considered as an attempt of further development of the Whitham approach [1]. G. B. Whitham investigated hyperbolic systems and causality relations between the reduced and original systems, whereas in our parabolic examples we are looking for ill posedness of the initial value problem for the reduced systems.

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